Composition of Chaotic Maps with an Invariant Measure

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Abstract

We generate new hierarchy of many-parameter family of maps of the interval $[0, 1]$ with an invariant measure, by composition of the chaotic maps of reference [1]. Using the measure, we calculate Kolmogorov-Sinai entropy, or equivalently Lyapunov characteristic exponent, of these maps analytically, where the results thus obtained have been approved with numerical simulation. In contrary to the usual one-dimensional maps and similar to the maps of reference [1], these maps do not possess period doubling or period-n-tupling cascade bifurcation to chaos, but they have single fixed point attractor at certain region of parameters values, where they bifurcate directly to chaos without having period-n-tupling scenario exactly at these values of parameter whose Lyapunov characteristic exponent begins to be positive.

Keywords:Chaos, Invariant measure, Entropy, Lyapunov characteristic exponent, Ergodic dynamical systems .

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1 INTRODUCTION

In the past twenty years dynamical systems, particularly one dimensional iterative maps have attracted much attention and have become an important area of research activity. One of the landmarks in it was introduction of the concept of Sinai-Ruelle-Bowen (SRB) measure or natural invariant measure[2, 3]. This is, roughly speaking, a measure that is supported on an attractor and also describe the statistics of the long time behavior of the orbits for almost every initial condition in the corresponding basin of attractor . This measure can be obtained by computing the fixed density of the so called Frobenius-Perron operator which can be viewed as a differential-integral operator, hence, exact determination of invariant measure of dynamical systems is rather a nontrivial task, such that invariant measure of few dynamical systems such as one-parameter family one-dimensional piecewise linear maps [4, 5] including Baker and tent maps or unimodal maps such as logistic map for certain values of its parameter, can be derived analytically. In most cases only numerical algorithms, as an example Ulam's method[6, 7, 8] are used for computation of fixed densities of Frobenius-Perron operator.

Authors in reference [1] have given hierarchy of one parameter family of nonlinear maps of interval $[0, 1]$ with an invariant measure. Here in this paper we generate new hierarchy of many-parameter family of maps of the interval $[0, 1]$ with an invariant measure, by composition of the chaotic maps of reference [1]. These maps are also defined as ratio of polynomials, where we have derived analytically their invariant measure for arbitrary values of the parameters. Using this measure, we have calculated analytically, Kolmogorov-Sinai entropy or equivalently positive Lyapunov characteristic exponent of these maps, where the numerical simulation approve the analytic calculation. Also it is shown that just like the maps of reference [1], they possess very peculiar property, that is, contrary to the usual, these maps do not possess period doubling or period-n-tupling cascade bifurcation to chaos, but instead they have single fixed point attractor at certain region of parameters values,

where they bifurcate directly to chaos without having period-n-tupling scenario exactly at these values of parameter whose Lyapunov characteristic exponent begins to be positive. The paper is organized as follows: In section II we introduce new hierarchy of family of many-parameters maps by composition of the chaotic maps of refernce [1]. In Section III we show that the proposed anzats for the invariant measure of these maps are eigenfuntion of Ferobenios-Perron operator with largest eigenvalue 1. Then in section IV using this measure we calculate Kolmogorov-Sinai entropy of these maps for an arbitrary value of parameters. In section V we compare analytic calculation with the numerical simulation. Paper ends with a brief conclusion.

2 Many-Parameter Families of Chaotic Maps

Let us consider the one-parameter families of chaotic maps of the interval $[0,1]$ given in reference [1], defined as the ratio of polynomials of degree N:

$$\begin{aligned}\Phi_N(x,\alpha) &= \frac{\alpha^2\left(1+(-1)^N {}_2F_1(-N,N,\frac{1}{2},x)\right)}{(\alpha^2+1)+(\alpha^2-1)(-1)^N {}_2F_1(-N,N,\frac{1}{2},x)} \\ &= \frac{\alpha^2(T_N(\sqrt{x}))^2}{1+(\alpha^2-1)(T_N(\sqrt{x})^2)} \quad ,\end{aligned} \tag{2-1}$$

where N is an integer greater than one. Also

$$_2F_1(-N,N,\frac{1}{2},x) = (-1)^N \cos\left(2N \arccos\sqrt{x}\right) = (-1)^N T_{2N}(\sqrt{x})$$

is hypergeometric polynomials of degree N and $T_N(x)(U_N(x))$ are Chebyshev polynomials of type I (type II)[9], respectively. Obviously these map the unit interval $[0,1]$ into itself. $\Phi_N(x,\alpha)$ is (N-1)-model map, that is it has $(N-1)$ critical points in unit interval $[0,1]$,(see Figure 4) since its derivative is proportional to derivative of hypergeometric polynomial $_2F_1(-N,N,\frac{1}{2},x)$ which is itself a hypergeometric polynomial of degree $(N-1)$, hence it has $(N-1)$ real roots in unit interval $[0,1]$. Defining Shwarzian derivative[10] $S\Phi_N(x,\alpha)$ as:

$$S\left(\Phi_N(x,\alpha)\right) = \frac{\Phi_N'''(x,\alpha)}{\Phi_N'(x,\alpha)} - \frac{3}{2}\left(\frac{\Phi_N''(x,\alpha)}{\Phi_N'(x,\alpha)}\right)^2 = \left(\frac{\Phi_N''(x,\alpha)}{\Phi_N'(x,\alpha)}\right)' - \frac{1}{2}\left(\frac{\Phi_N''(x,\alpha)}{\Phi_N'(x,\alpha)}\right)^2,$$

with a prime denoting a single differentiation with respect to variable x, one can show that[1]:

$$S\left(\Phi_N(x,\alpha)\right) = S\left({}_2F_1(-N,N,\frac{1}{2},x)\right) \leq 0.$$

Therefore, the maps $\Phi_N(x)$ have at most $N+1$ attracting periodic orbits[10]. As it is shown in reference [1], these maps have only single period one stable fixed points.

Using the above hierarchy of family one-parameter of maps we can generate new hierarchy of family many-parameters chaotic maps with an invariant measure simply from the composition of these maps. Hence considering the functions $\Phi_{N_k}(x,\alpha_k), k=1,2,\cdots,n$ we denote their composition by: $\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x)$ which can be written in terms of them in the following form:

$$\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x) = \overbrace{(\Phi_{N_1}\circ\Phi_{N_2}\circ\cdots\circ\Phi_{N_n}(x))}^{n} =$$

$$\Phi_{N_1}(\Phi_{N_2}(\cdots(\Phi_{N_n}(x,\alpha_n),\alpha_{(n-1)})\cdots,\alpha_2),\alpha_1) \qquad \text{(2-2)}$$

Since these maps consist of the composition of the (N_k-1)-modals $(k=1,2,\cdots,n)$ maps with negative Shwarzian derivative, therefore, they are $(N_1N_2\cdots N_n-1)-$ modals map and their Shwarzian derivative is negative too [10]. Therefore these maps have at most $N_1N_2\cdots N_n+1$ attracting periodic orbits[10]. As we will show below in this section, these maps have only a single period one stable fixed points. Since, denoting m-composition of these functions by $\Phi^{(m)}$, it is straightforward to show that the derivative of $\Phi^{(m)}$ at its possible $m\times n$ periodic points of an m-cycle: $x_{\mu,k+1} = \Phi_{N_k}(x_{\mu,k},\alpha_k), x_{1,\mu+1} = \Phi_{N_n}(x_{n,\mu},\alpha_N), \mu = 1,2\cdots,m$ and $k=1,2,\cdots,n$ and $x_{1,1} = \Phi_{N_n}(x_{m,n},\alpha_n)$ is

$$\mid \frac{d}{dx}\Phi^{(m)} \mid == \prod_{\mu=1}^{m}(\prod_{k=1}^{n} \mid \frac{N_k}{\alpha_k}(\alpha_k^2+(1-\alpha_k^2)x_{\mu,k}) \mid, \qquad \text{(2-3)}$$

since for $x_{\mu,k}\in[0,1]$ we have:

$$min(\alpha_k^2+(1-\alpha_k^2 x_{\mu,k})) = min(1,\alpha_k^2),$$

therefore,

$$min \mid \frac{d}{dx}\Phi^{(m)} \mid = \prod_{k=1}^{n}\left(\frac{N_k}{\alpha_k}min(1,\alpha_k^2)\right)^m.$$

Hence the above expression is definitely greater than one for $\prod_{k=1}^{n}\frac{1}{N_k} < \prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}N_k$, that is, these maps do not have any kind of m-cycle or periodic orbits in the region of the parameters space defined by $\prod_{k=1}^{n}\frac{1}{N_k} < \prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}N_k$, actually they are ergodic in this region of the parameters space. From (2-3) it follows that $\mid \frac{d}{dx}\Phi^{(m)} \mid$ at $m \times n$ periodic points of the m-cycle belonging to interval [0,1], varies between $\prod_{k=1}^{n}(N_k\alpha_k)^m$ and $\prod_{k=1}^{n}(\frac{N_k}{\alpha_k})^m$ for $\prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}\frac{1}{N_k}$ and between $\prod_{k=1}^{n}(\frac{N_k}{\alpha_k})^m$ and $\prod_{k=1}^{n}(N_k\alpha_k)^m$ for $\prod_{k=1}^{n}\alpha_k > \prod_{k=1}^{n}N_k$, respectively.

From the definition of these maps, we see that definitely $x = 1$ and $x = 0$ (in special case of odd integer values of $N-1, N_2, \cdots, N_n$) belong to one of the m-cycles.

For $\prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}\frac{1}{N_k}(\prod_{k=1}^{n}\alpha_k > \prod_{k=1}^{n}N_k)$, the formula $(2-3)$ implies that for those cases in which $x = 1(x = 0)$ belongs to one of m-cycles we will have $\mid \frac{d}{dx}\Phi^{(m)} \mid < 1$, hence the curve of $\Phi^{(m)}$ starts at $x = 1(x = 0)$ beneath the bisector and then crosses it at the previous (next) periodic point with slope greater than one (see Fig. 1), since the formula $(2-3)$ implies that the slope of fixed points increases with the decreasing (increasing) of $\mid x_{\mu,k} \mid$, therefore at all periodic points of n-cycles except for $x = 1(x = 0)$ the slope is greater than one that is they are unstable, this is possible only if $x = 1(x = 0)$ is the only period one fixed point of these maps.

Hence all m-cycles except for possible period one fixed points $x = 1$ and $x = 0$ are unstable. Actually, the fixed point $x = 0$ is the stable fixed point of these maps in the regions of the parameters spaces defined by $\alpha_k > 0, k = 1, 2, \cdots, n$ and $\prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}\frac{1}{N_k}$ only for odd integer values of $N_1, N_2, \cdots, N_n$, however, if one of the integers $N_k, k = 1.2, \cdots, n$ happens to be even, then the $x = 0$ will not be a stable fixed point anymore. But, the fixed point $x = 1$ is stable fixed point of these maps in the regions of the parameters spaces defined by $\prod_{k=1}^{n}\alpha_k > \prod_{k=1}^{n}N_k$ and $\alpha_k < \infty, k = 1, 2, \cdots, n$ for all integer values of $N_1, N_2, \cdots, N_n$.

As an example we give below some of these maps:

$$\phi_{2,2}^{\alpha_1,\alpha_2}(x) = \frac{{\alpha_1}^2\left(4\,x\,(x-1)+(2\,x-1)^2\,{\alpha_2}^2\right)^2}{{\alpha_1}^2\left(4\,x\,(x-1)+(2\,x-1)^2\,{\alpha_2}^2\right)^2+h1} \tag{2-4}$$

$$\phi_{2,3}^{\alpha_1,\alpha_2}(x) = \frac{{\alpha_1}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_2}^2\right)^2}{{\alpha_1}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_2}^2\right)^2+h2} \tag{2-5}$$

$$\phi_{3,2}^{\alpha_1,\alpha_2}(x) = \frac{{\alpha_2}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_1}^2\right)^2}{{\alpha_2}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_1}^2\right)^2+h3} \tag{2-6}$$

$$\phi_{3,3}^{\alpha_1,\alpha_2}(x) = \frac{{\alpha_1}^2{\alpha_2}^2x\,(4\,x-3)^2\left(3\;(x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_2}^2\right)}{-\,(x-1)^3\,(4\,x-1)^6+3\,x\,(3\,{\alpha_1}^2-2)\,(4\,x-3)^2\,(x-1)^2\,(4\,x-1)^4\,{\alpha_2}^2+h4} \tag{2-7}$$

where :

$$h1 = -16\,{\alpha_2}^2(2\,x-1)^2x(x-1)$$

$$h2 = -4\,x(x-1)(4\,x-1)^2(4\,x-3)^2{\alpha_2}^2$$

$$h3 = 4\,x(x-1)(4\,x-1)^2(4\,x-3)^2{\alpha_1}^2$$

$$h4 = 3\,x^2\left(-3+2\,{\alpha_1}^2\right)(x-1)\,(4\,x-1)^2\,(4\,x-3)^4\,{\alpha_2}^4+{\alpha_1}^2x^3\,(4\,x-3)^6\,{\alpha_2}^6$$

Below we also introduce their conjugate or isomorphic maps which will be very useful in derivation of their invariant measure and calculation of their KS-entropy in the next section. Conjugacy means that the invertible map $h(x) = \frac{1-x}{x}$ maps $I = [0,1]$ into $[0,\infty)$ and transforms maps $\Phi_{N_k}(x,\alpha_k)$ into $\tilde{\Phi}_{N_k}(x,\alpha_k)$ defined as:

$$\tilde{\Phi}_{N_k}(x,\alpha_k) = h\circ\Phi_{N_k}(x,\alpha_k)\circ h^{(-1)} = \frac{1}{\alpha_k^2}\tan^2(N_k\arctan\sqrt{x})$$

Hence this transforms the maps $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)$ into $\tilde{\Phi}_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)$ defined as:

$$\tilde{\Phi}^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x)$$
$$\frac{1}{\alpha_1^2}\tan^2(N_1\arctan\sqrt{\circ}\frac{1}{\alpha_2^2}\tan^2(N_2\arctan\sqrt{\circ}\cdots\circ\ \frac{1}{\alpha_n^2}\tan^2(N_n\arctan\sqrt{x})))$$
$$\frac{1}{\alpha_1^2}\tan^2(N_1\arctan\sqrt{\frac{1}{\alpha_2^2}\tan^2(N_2\arctan\sqrt{\cdots\frac{1}{\alpha_n^2}\tan^2(N_n\arctan\sqrt{x})}\cdots))} \tag{2-8}$$

3 INVARIANT MEASURE

Dynamical systems, even apparently simple dynamical systems as those described by maps of an interval, can display a rich variety of different asymptotic behavior. On measure theoretical level these types of behavior are described by SRB [2] or invariant measure describing statistically stationary states of the system. The probability measure μ on $[0,1]$ is called an SRB or invariant measure of the maps $\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x)$ given in $(2-2)$, if it is $\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x)$-invariant and absolutely continuous with respect to Lebesgue measure. For deterministic system such as these composed maps, the $\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x)$-invariance means that its invariant measure $\mu_{\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}}(x)$ fulfills the following formal Ferbenius-Perron integral equation

$$\mu_{\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}}(y)=\int_0^1\delta(y-\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x))\mu_{\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}}(x)dx.$$

This is equivalent to:

$$\mu_{\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}}(y)=\sum_{x\in\ \Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(y)}\mu_{\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}}(x)\frac{dx}{dy}\quad, \tag{3-1}$$

defining the action of standard Ferobenius-Perron operator for the map $\Phi(x)$ over a function as:

$$P_{\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}}f(y)=\sum_{x\in\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(y)}f(x)\frac{dx}{dy}\quad. \tag{3-2}$$

We see that, the invariant measure $\mu_{\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}}(x)$ is given as the eigenstate of the Frobenius-Perron operator $P_{\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}}$ corresponding to the largest eigenvalue 1.

As we will prove below the measure $\mu_{\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}}(x,\beta)$ defined as:

$$\frac{1}{\pi}\frac{\sqrt{\beta}}{\sqrt{x(1-x)}(\beta+(1-\beta)x)} \quad , \tag{3-3}$$

is the invariant measure of the maps $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)$ provided that the parameter β is positive and fulfills the following relation:

$$\prod_{k=1}^{n}\alpha_k \times \frac{A_{N_n}(\frac{1}{\beta})}{B_{N_n}(\frac{1}{\beta})} \times \frac{A_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})} \times \frac{A_{N_{n-2}}(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-2}}(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})})} \times \cdot \times \frac{A_{N_1}(\frac{1}{\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,\alpha_n}(\frac{1}{\beta})})}{B_{N_1}(\frac{1}{\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,\alpha_n}(\frac{1}{\beta})})} = 1 \tag{3-4}$$

where the polynomials $A_{N_k}(x)$ and $B_{N_k}(x)$ $(k=1,2,\cdots,n)$ are defined as:

$$A_{N_k}(x) = \sum_{l=0}^{[\frac{N_k}{2}]} C_{2l}^{N_k} x^l,$$

$$B_{N_k}(x) = \sum_{l=0}^{[\frac{N_k-1}{2}]} C_{2l+1}^{N_k} x^l,$$

where [ ] means greatest integer part. Also the functions $\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})$, $\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta}),\cdots$ and $\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,\alpha_n}(\frac{1}{\beta})$ are defined in the following form:

$$\eta_{N_n}^{\alpha_n}(\frac{1}{\beta}) = \beta(\frac{\alpha_n A_{N_n}(\frac{1}{\beta})}{B_{N_n}(\frac{1}{\beta})})^2$$

$$\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta}) = \beta(\frac{\alpha_{n-1} A_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})})^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,\alpha_n}(\frac{1}{\beta}) = \beta(\frac{\alpha_2 A_{N_2}(\frac{1}{\eta_{N_3,N_4,\cdots,N_n}^{\alpha_3,\alpha_4,\cdots,\alpha_n}(\frac{1}{\beta})})}{B_{N_2}(\frac{1}{\eta_{N_3,N_4,\cdots,N_n}^{\alpha_3,\alpha_4,\cdots,\alpha_n}(\frac{1}{\beta})})})^2$$

As we see the above measure is defined only for $\beta > 0$ hence, from the relations $(3-4)$, it follows that these maps are ergodic in the region of the parameter space which leads to

positive solution of β. Taking the limits of $\beta \longrightarrow 0_+$ and $\beta \longrightarrow \infty$ in the relation (3-4),respectively, one can show that the ergodic regions are : $\prod_{k=1}^{n}\frac{1}{N_k} < \prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}N_k$ for odd integer values of $N_1, N_2, \cdots, N_n$ and $\alpha_k > 0, \mathrm{for} k = 1, 2, \cdots, n \& \prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}N_k$ if one of the integers happens to become even, respectively. Out of these regions they have only single stable fixed points.

In order to prove that measure $(3-3)$ satisfies equation $(3-1)$, with β given by relation $(3-4)$, it is rather convenient to consider the conjugate map:

$$\tilde{\Phi}_{N_1,N_2,\cdots,N_n}\alpha_1, \alpha_2, \cdots, \alpha_n(x),$$

with measure $\tilde{\mu}_{\tilde{\Phi}_{N_1,N_2\cdots,N_n}^{\alpha_1,\alpha_2\cdots,\alpha_n}}$ denoted by $\tilde{\mu}_{\tilde{\phi}}$ related to the measure $\mu_{\Phi_{N_1,N_2\cdots,N_n}^{\alpha_1,\alpha_2\cdots,\alpha_n}}$ denoted by μ_{ϕ} through the following relation:

$$\tilde{\mu_{\tilde{\Phi}}}(x) = \frac{1}{(1+x)^2}\mu_{\Phi}(\frac{1}{1+x}).$$

Denoting $\tilde{\Phi}_{N_1,N_2,\cdots,N_n}\alpha_1, \alpha_2, \cdots, \alpha_n(x)$ by y and inverting it, we get :

$$x_{k_1} = \tan^2(\frac{1}{N_1}\arctan\sqrt{y\alpha_1^2} + \frac{k_1\pi}{N_1}) \qquad k_1 = 1, .., N_1.$$

$$x_{k_1,k_2} = \tan^2(\frac{1}{N_2}\arctan\sqrt{x_{k_1}\alpha_2^2} + \frac{k_2\pi}{N_2}) \qquad k_2 = 1, .., N_2.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{k_1,k_2,\cdots,k_n} = \tan^2(\frac{1}{N_n}\arctan\sqrt{x_{k_1,k_2,\cdots,k_{n-1}}\alpha_n^2} + \frac{k_n\pi}{N_n}) \qquad k_1 = 1, .., N_1.$$

Then, taking derivative of $x_{k_1,k_2,\cdots,k_n}$ with respect to y, we obtain:

$$\mid \frac{dx_{k1,k_2,\cdots,k_n}}{dy} \mid = (\prod_{k=1}^{n}\frac{\alpha_k}{N_k})\sqrt{\frac{x_{k_1,k_2,\cdots,k_n}}{y}}$$

$$\frac{(1+x_{k_1,k_2,\cdots,k_n})(1+x_{k_2,k_3,\cdots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\cdots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\cdots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{k_n})(1+\alpha_1^2 y)}. \qquad (3\text{-}5)$$

In derivation of the formula we have used the chain rule properties of the derivative of composite functions.

Substituting the above result in equation $(3-1)$, we have:

$$\tilde{\mu}_{\tilde{\Phi}}(y)\sqrt{y}(1+\alpha_1^2 y) = (\prod_{k=1}^{n}\frac{\alpha_k}{N_k})\sum_{k_1}\sum_{k_2}\cdots\sum_{k_n}\sqrt{x_{k_1,k_2,\cdots,k_n}}$$

$$\times\frac{(1+x_{k_1,k_2,\cdots,k_n})(1+x_{k_2,k_3,\cdots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\cdots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\cdots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{k_n})}\tilde{\mu}_{\tilde{\Phi}}(x_{k_1,k_2,\cdots,k_n}).$$

Now,considering the following anzatz for the invariant measure $\tilde{\mu}_{\tilde{\Phi}}(y)$:

$$\tilde{\mu}_{\tilde{\Phi}}(y) = \frac{1}{\sqrt{y}(1+\beta y)} \quad , \tag{3-6}$$

then the above equation reduces to:

$$\frac{1+\alpha_1^2 y}{1+\beta y} = (\prod_{k=1}^{n}\frac{\alpha_k}{N_k})$$

$$\times\sum_{k_1=1}^{N_1}\sum_{k_2=1}^{n_2}\cdots\sum_{k_n=1}^{N_n}\left(\frac{(1+x_{k_1,k_2,\cdots,k_n})(1+x_{k_2,k_3,\cdots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\cdots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\cdots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{k_n})}\right).$$

Now, using the formula (3-5) of reference[1] we obtain:

$$\frac{\alpha_n}{N_n}\sum_{k_n=1}^{N_n}\frac{1+x_{k_1,k_2,\cdots,k_n}}{1+\beta x_{k_1,k_2,\cdots,k_n}} = \frac{\alpha_n A_{N_n}(\frac{1}{\beta})}{B_{N_n}(\frac{1}{\beta})}\frac{1+\alpha_n^2 x_{k_1,k_2,\cdots,k_{n-1}}}{1+\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})x_{k_1,k_2,\cdots,k_{n-1}}},$$

$$\frac{\alpha_{n-1}\alpha_n}{N_{n-1}N_n}\sum_{k_{n-1}=1}^{N_{n-1}}\sum_{k_n=1}^{N_n}\frac{(1+x_{k_1,k_2,\cdots,k_{n-1}})(1+x_{k_1,k_2,\cdots,k_n})}{(1+\alpha_{n-1}x_{k_1,k_2,\cdots,k_{n-1}})(1+\beta x_{k_1,k_2,\cdots,k_n})} =$$

$$\frac{\alpha_n\alpha_{n-1}A_{N_n}(\frac{1}{\beta})A_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}{B_{N_n}(\frac{1}{\beta})B_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}\times\frac{1+\alpha_{n-1}^2 x_{k_1,k_2,\cdots,k_{n-2}}}{1+\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})x_{k_1,k_2,\cdots,k_{n-2}}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\prod_{k=1}^{n}\frac{\alpha_k}{N_k})\sum_{k_1=1}^{N_1}\sum_{k_2=1}^{n_2}\cdots$$

$$\times\sum_{k_n=1}^{N_n}\left(\frac{(1+x_{k_1,k_2,\cdots,k_n})(1+x_{k_2,k_3,\cdots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\cdots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\cdots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{kn})}\right) =$$

$$\prod_{k=1}^{n}\alpha_k \times \frac{A_{N_n}(\frac{1}{\beta})}{B_{N_n}(\frac{1}{\beta})} \times \frac{A_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}$$

$$\times\frac{A_{N_{n-2}}(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-2}}(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})})} \times \cdot \times \frac{A_{N_1}(\frac{1}{\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,a_n}(\frac{1}{\beta})})}{B_{N_1}(\frac{1}{\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,\alpha_n}(\frac{1}{\beta})})}\frac{1+\alpha_1^2 y}{1+\eta_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(\frac{1}{\beta})y}.$$

Now, inserting the right hand side of last relation in $(3-5)$, we get:

$$\frac{1+\alpha_1^2 y}{1+\beta y} = \prod_{k=1}^{n}\alpha_k \frac{A_{N_n}(\frac{1}{\beta})}{B_{N_n}(\frac{1}{\beta})} \times \frac{A_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-1}}(\frac{1}{\eta_{N_n}^{\alpha_n}(\frac{1}{\beta})})}\times$$

$$\frac{A_{N_{n-2}}(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})})}{B_{N_{n-2}}(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}(\frac{1}{\beta})})} \times \cdot \times \frac{A_{N_1}(\frac{1}{\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,a_n}(\frac{1}{\beta})})}{B_{N_1}(\frac{1}{\eta_{N_2,N_3,\cdots,N_n}^{\alpha_2,\alpha_3,\cdots,\alpha_n}(\frac{1}{\beta})})}\frac{1+\alpha_1^2 y}{1+\eta_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(\frac{1}{\beta})y}.$$

We see that the above relation will hold true provided that the parameter β fulfills the relation (3-4).

4 KOLMOGROV-SINAI ENTROPY

Kolomogrov-Sinai entropy (KS) or metric entropy [2] measure how chaotic a dynamical system is and it is proportional to the rate at which information about the state of dynamical system is lost in the course of time or iteration. Therefore, it can also be defined as the average rate of information loss for a discrete measurable dynamical system $(\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x),\mu)$, by introducing a partition $\alpha = A_c(n_1,.....n_\gamma)$ of the interval $[0,1]$ into individual laps A_i one can define the usual entropy associated with the partition by:

$$H(\mu,\gamma) = -\sum_{i=1}^{n(\gamma)} m(A_c)\ln m(A_c),$$

where $m(A_c) = \int_{n\in A_i}\mu(x)dx$ is the invariant measure of A_i. Defining n-th refining $\gamma(n)$ of γ:

$$\gamma^n = \bigcup_{k=0}^{n-1}(\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x))^{-(k)}(\gamma),$$

and defining an entropy per unit step of refining by :

$$h(\mu,\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x),\gamma) = \lim{}_{n\to\infty}(\frac{1}{n}H(\mu,\gamma)),$$

if the size of individual laps of $\gamma(N)$ tends to zero as n increases, then the above entropy is known as Kolmogorov-Sinai entropy, that is:

$$h(\mu,\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)) = h(\mu,\Phi N_1,N_2,\cdots,N_{n(}^{\alpha_1,\alpha_2,\cdots,\alpha_n}x),\gamma).$$

KS-entropy , which is a quantitative measure of the rate of information loss with the refining, may also be written as:

$$h(\mu,\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)) = \int\mu(x)dx\ln|\frac{d}{dx}\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)|, \tag{4-1}$$

which is also a statistical mechanical expression for the Lyapunov characteristic exponent, that is, the mean divergence rate of two nearby orbits. The measurable dynamical system $(\Phi N_1,N_2,\cdots,N_{n(}^{\alpha_1,\alpha_2,\cdots,\alpha_n}x),\mu)$ is chaotic for $h>0$ and predictive for $h=0$.

In order to calculate the KS-entropy of the maps $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)$, it is rather convenient to consider their conjugate maps given by $(2-8)$, since it can be shown that KS-entropy is a kind of topological invariant, that is, it is preserved under conjugacy map. Hence we have:

$$h(\mu,\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)) = h(\tilde{\mu},\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\tilde{\alpha_2},\cdots,\alpha_n}(x)).$$

Using the integral $(4-1)$, the KS-entropy of $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)$ can be written as

$$h(\mu,\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)) = h(\tilde{\mu},\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\tilde{\alpha_2},\cdots,\alpha_n}(x)) =$$
$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(|\frac{d}{dy_{N_2,N_3,\cdot,N_n}}(\frac{1}{\alpha_1^2}\tan^2(N_1\arctan\sqrt{y_{N_2,N_3,\cdot,N_n}}))\times$$

$$\frac{d}{dy_{N_3,N_4,\cdot,N_n}}(\frac{1}{\alpha_2^2}\tan^2(N_2\arctan\sqrt{y_{N_3,N_4,\cdot,N_n}}))\cdots\frac{d}{dx}(\frac{1}{\alpha_n^2}\tan^2(N_n\arctan\sqrt{x}))\mid)$$

or

$$h(\mu,\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x)) =$$
$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(\mid\frac{d}{dy_{N_2,N_3,\cdot,N_n}}(\frac{1}{\alpha_1^2}\tan^2(N_1\arctan\sqrt{y_{N_2,N_3,\cdot,N_n}})))+$$
$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(\mid\frac{d}{dy_{N_3,N_4,\cdot,N_n}}(\frac{1}{\alpha_2^2}\tan^2(N_2\arctan\sqrt{y_{N_3,N_4,\cdot,N_n}})))+\cdots+$$
$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(\mid\frac{d}{dy_{N_n}}(\frac{1}{\alpha_{n-1}^2}\tan^2(N_{n-1}\arctan\sqrt{y_{N_n}})))+$$
$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(\mid\frac{d}{dx}(\frac{1}{\alpha_n^2}\tan^2(N_n\arctan(\sqrt{x}))))\mid) \tag{4-2}$$

where

$$y_{N_n} = \frac{1}{\alpha_n^2}\tan^2(N_n\arctan(\sqrt{x})) \tag{4-3}$$
$$y_{N_{n-1},N_n} = \frac{1}{\alpha_{n-1}}\tan^2(N_{n-1}\arctan(\sqrt{y_{N_n}})) \tag{4-4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_{N_2,N_3,\cdot,N_n} = \frac{1}{\alpha_1^2}\tan^2(N_1\arctan(\sqrt{y_{N_3,N_4,\cdot,N_n}})). \tag{4-5}$$

. Now, we calculate the integrals appearing above in the expression for the entropy, separately. The last integral in (4-2) can be calculated with the prescription of section IV of the reference [1] and it reads:

$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(\mid\frac{d}{dx}(\frac{1}{\alpha_n^2}\tan^2(N_n\arctan(\sqrt{x}))))\mid)=\ln\left(\frac{N_n(1+\beta+2\sqrt{\beta})^{N_n-1}}{A_{N_n}(\frac{1}{\beta})B_{N_n}(\frac{1}{\beta})}\right). \tag{4-6}$$

In order to calculate the integral one before last in (4-2), that is:

$$\frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(|\frac{d}{dy_{N_n}}(\frac{1}{\alpha_{n-1}^2}\tan^2(N_{n-1}\arctan\sqrt{y_{N_n}}))) \qquad (4\text{-}7)$$

first we make the following change of variable by inverting the relation(4-3):

$$x_{k_{n-1}} = \tan^2(\frac{1}{N_{n-1}}\arctan(\sqrt{y_{N_n}}\alpha_{n-1}^2)+\frac{k_{n-1}\pi}{N_{n-1}}) \qquad k_{n-1}=1,..,N_{n-1}.$$

then the integral (4-7) reduces to:

$$\sum_{k_{n-1}=1}^{N_{n-1}}\frac{1}{\pi}\int_{x^i_{k_{n-1}}}^{x^f_{k_{n-1}}}\frac{\sqrt{\beta}dx_{k_{n-1}}}{\sqrt{x_{k_{n-1}}}(1+\beta x_{k_{n-1}})}\ln(|\frac{d}{dy_{N_n}}(\frac{1}{\alpha_{n-1}^2}\tan^2(N_{n-1}\arctan\sqrt{y_{N_n}}))$$

where $x^i_{k_{n-1}}$ and $x^f_{k_{n-1}}$ ($k_{n-1}=1,2,\cdots,N_{n-1}$) denote the initial and end points of k-th branch of the inversion of function $y_{N_n}=(\frac{1}{\alpha_n^2}\tan^2(N_n\arctan\sqrt{x}))$, respectively. Now, inserting the derivative of $x_{k_{n-1}}$ with respect to y_{N_n} in the above relation and changing the order of sum and integration, we get:

$$\frac{1}{\pi}\int_0^\infty\sum_{k_{n-1}=1}^{N_{n-1}}\sqrt{\beta}dy_{N_n}\frac{\alpha_{n-1}\sqrt{x_{k_{n-1}}}(1+x_{k_{n-1}})}{N_{n-1}\sqrt{y_{n_n}}(1+\alpha_{n-1}^2 y_{N_n})\sqrt{x_{k_{n-1}}}(1+\beta x_{k_{n-1}})}$$

$$\times\ln(|\frac{d}{dy_{N_n}}(\frac{1}{\alpha_{n-1}^2}\tan^2(N_{n-1}\arctan\sqrt{y_{N_n}}))).$$

Using the formula (3-5) of reference [1], it reduces to

$$\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dy_n}{\sqrt{y_n}}(\frac{B_{N_n}(\frac{1}{\beta})}{\alpha_n A_{N_n}(\frac{1}{\beta})}+\beta\frac{\alpha_n A_{N_n}(\frac{1}{\beta})}{B_{N_n}(\frac{1}{\beta})}y_{N_n})\ln(|\frac{d}{dy_{N_n}}(\frac{1}{\alpha_{n-1}^2}\tan^2(N_{n-1}\arctan\sqrt{y_{N_n}}))).$$

Finally, calculating the above integral with the prescription of reference[1] we obtain:

$$\ln\left(\frac{N_{n-1}(1+\eta_{N_n}^{\alpha_n}+2\sqrt{\eta_{N_n}^{\alpha_n}})^{N_{n-1}-1}}{A_{N_{n-1}}(\eta_{N_n}^{\alpha_n})B_{N_{n-1}}(\eta_{N_n}^{\alpha_n})}\right).$$

Similarly, we can calculate the other integrals appearing in the expression for the entropy of the composed maps given in (4-2):

$$=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln(|\frac{d}{dy_{N_k,N_{k+1},\cdot,N_n}}(\frac{1}{\alpha_{k-1}^2}\tan^2(N_{k-1}\arctan\sqrt{y_{N_k,N_{k+1},\cdot,N_n}}))=$$

$$\ln\left(\frac{N_{k-1}(1+\eta^{\alpha_k,\alpha_{k+1},\cdots,\alpha_n}_{N_k,N_{k-1},\cdots,N_n}(\frac{1}{\beta})+2\sqrt{\eta^{\alpha_k,\alpha_{k+1},\cdots,\alpha_n}_{N_k,N_{k-1},\cdots,N_n}(\frac{1}{\beta})})^{N_{k-1}-1}}{A_{N_{k-1}}(\eta^{\alpha_k,\alpha_{k+1},\cdots,\alpha_n}_{N_k,N_{k-1},\cdots,N_n}(\frac{1}{\beta}))B_{N_{k-1}}(\eta^{\alpha_k,\alpha_{k+1},\cdots,\alpha_n}_{N_k,N_{k-1},\cdots,N_n})(\frac{1}{\beta})}\right).\text{for}k=1,2,\cdots,n$$

Finally summing the above integral we get the following expression for the entropy of these maps:

$$h(\mu,\Phi^{\alpha_1,\alpha_2,\cdots,\alpha_n}_{N_1,N_2,\cdots,N_n}(x))=$$

$$\ln(\frac{(N_1N_2\cdots N_n)(1+\sqrt{\beta})^{2(N_n-1)}(1+\sqrt{\eta^{\alpha_n}_{N_n}(\frac{1}{\beta})})^{2(N_{n-1}-1)}\cdots(1+\sqrt{\eta^{\alpha_2,\alpha_3,\cdots,\alpha_n}_{N_2,N_3,\cdots,N_n}(\frac{1}{\beta})})^{2(N_1-1)}}{A_{N_n}(\beta)B_{N_n}(\beta)A_{N_{n-1}}(\eta^{\alpha_n}_{N_n}(\frac{1}{\beta}))B_{N_{n-1}}(\eta^{\alpha_n}_{N_n}(\frac{1}{\beta}))\cdots A_{N_1}(\eta^{\alpha_2,\alpha_3,\cdots,\alpha_n}_{N_2,N_3,\cdots,N_n}(\frac{1}{\beta}))B_{N_1}(\eta^{\alpha_2,\alpha_3,\cdots,\alpha_n}_{N_2,N_3,\cdots,N_n}(\frac{1}{\beta}))}). \tag{4-8}$$

Imposing the relations between the parameters $\alpha_k, k=1,2,\cdots,n$ which are consistent with the relation (3-4), reduces these maps to other maps of many-parameters family of the maps with number of the parameters less than n. Particularly by imposing enough relations we can reduce them to one-parameter family of chaotic maps with an arbitrary asymptotic behavior as the parameter takes the limiting values. Hence we can construct chaotic maps with arbitrary universality class. As an illustration we consider the chaotic map $\Phi^{\alpha_1,\alpha_2}_{2,2}(x)$. Using the formula (4-8) we have

$$h(\mu,\Phi^{\alpha_1,\alpha_2}_{2,2}(x)=\ln\frac{(1+\sqrt{\beta})^2(2\sqrt{\beta}+\alpha_2(1+\beta))^2}{(1+\beta)(4\beta+\alpha_2^2(1+\beta)^2)}$$

with the following relation among the parameters α_1,α_2 and β:

$$\alpha_1(4\beta+\alpha_2^2(1+\beta)^2)=4\alpha_2\beta(1+\beta)$$

which is obtained from the relation(3-4). Now choosing $\beta=\alpha_2^{\nu}$, ,$0<\nu<2$, the above relation reduces to:

$$a_1=\frac{4\alpha_2^{1+\nu}(1+\alpha_2^{\nu})}{\alpha_2^2(1+\alpha_2^{\nu})^2+4\alpha_2^{\nu}}$$

and entropy given by (4-8) reads:

$$h(\mu,\Phi^{\alpha_2}_{2,2}(x)=\ln\frac{(1+\alpha_2^{\frac{\nu}{2}})^2(2\alpha_2^{\frac{\nu}{2}}+\alpha_2(1+\alpha_2^{\nu}))^2}{(1+\alpha_2^{\nu})(4\alpha_2^{\nu}+\alpha_2^2(1+\alpha_2^{\nu})^2)}$$

which has the following asymptotic behavior near $\alpha_2 \longrightarrow 0$ and $\alpha_2 \longrightarrow \infty$:

$$\begin{cases} h(\mu, \Phi_{2,2}^{\alpha_2}(x) \sim \alpha_2^{\frac{\nu}{2}} & \text{as} \alpha_2 \longrightarrow 0 \\ h(\mu, \Phi_{2,2}^{\alpha_2}(x) \sim (\frac{1}{\alpha_2})^{\frac{\nu}{2}} & \text{as} \alpha_2 \longrightarrow \infty. \end{cases}$$

The above asymptotic form indicates that, for an arbitrary value of $0 < \nu < 2$, the maps $\Phi_{2,2}^{\alpha_2}(x)$ belong to the universality class which is different from the universality class of chaotic maps of the reference[1] or the universality class of pitch fork bifurcating maps.

Here in this section we try to calculate Lyapunov characteristic exponent of maps $\Phi_N^{(1,2)}(x,\alpha)$, $N = 1, 2, ..., 5$ in order to investigate these maps numerically. In fact, Lyapunov characteristic exponent is the characteristic exponent of the rate of average magnificent of the neighborhood of an arbitrary point x_0 and it is denoted by $\Lambda(x_0)$ which is written as:

$$\begin{aligned} \Lambda_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_0) &= lim_{n\to\infty} \ln(| \overbrace{\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x,\alpha) \circ \Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n} .... \circ \Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}}^{n} | \\ &= lim_{n\to\infty} \sum_{k=0}^{n-1} \ln | \frac{d\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_k,\alpha)}{dx} |, \end{aligned} \tag{4-1}$$

where $x_k = \overbrace{\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n} \circ \Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n} \circ .... \circ \Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}}$. It is obvious that $\Lambda^{(1,2)}(x_0) < 0$ for an attractor, $\Lambda N_1, N_2, \cdots, {N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_0) > 0$ for a repeller and $\Lambda N_1, N_2, \cdots, {N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_0) = 0$ for marginal situation. Also the Liapunov number is independent of initial point x_0, provided that the motion inside the invariant manifold is ergodic, thus $\Lambda N_1, N_2, \cdots, {N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_0)$ characterizes the invariant manifold of $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}$ as a whole. For values of parameter $\alpha_k, k = 1, 2, \cdots, n$, such that the map $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}$ be measurable, Birkohf ergodic theorem implies equality of KS-entropy and Liapunov characteristic exponent, that is:

$$h(\mu, \Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}) = \Lambda_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x_0, \Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}). \tag{4-2}$$

Comparison of analytically calculated KS-entropy of maps $\Phi_{N_1,N_2,\cdots,N_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n}(x,\alpha)$ for $N_1 = 2, 3$ and $N_2 = 2, 3$, (see Figures $5, 6$ and 7) with the corresponding Lyapunov characteristic

exponent obtained by simulation, indicate that in chaotic region, these maps are ergodic as predicted by Birkohf ergodic theorem. In non chaotic region of the parameter, Lyapunov characteristic exponent is negative, since in this region we have only stable period one fixed points without bifurcation. In summary, combining the analytic discussion of section II with the numerical simulation we deduce that these maps are ergodic in certain region of their parameters space as explained above and in the complementary region of the parameters space they have only a single period one attractive fixed point, such that in contrary to the most of usual one-dimensional one-parameter or many-parameters family of maps they have only a bifurcation from a period one attractive fixed point to chaotic state or vice-versa.

5 Conclusion

We have given hierarchy of exactly solvable many-parameter family of one-dimensional chaotic maps with an invariant measure, that is measurable dynamical system with an interesting property of being either chaotic (proper to say ergodic) or having stable period one fixed point and they bifurcate from a stable single periodic state to chaotic one and vice-versa without having usual period doubling or period-n-tupling scenario.

Again this interesting property is due to existence of invariant measure for a region of the parameters space of these maps. Hence, to approve this conjecture, it would be interesting to find other measurable one parameter maps, specially higher dimensional maps, which is under investigation.

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Figures Captions

Fig.1. Plot of $\Phi_{2,2}^{\alpha_1,\alpha_2}(x)$ map for $\alpha_1 = 1.5, \alpha_2 = 2.9$ (curve with red color) and plot of $\Phi_{2,2}^{\alpha_1,\alpha_2} \circ \Phi_{2,2}^{\alpha_1,\alpha_2}(x)$ map for $\alpha_1 = 1.5, \alpha_2 = 2.9$, (curve with green color). The values of the maxima and minima (0 or 1) are independent from the parameters α_1, α_2, as shown in the figure.

Fig.2. Solid red surface shows the variation of KS-entropy $\Phi_{2,2}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2, dotted green surface curve shows variation of Lyapunov characteristic exponent of $\Phi_{2,2}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2.

Fig.3. Solid red surface shows the variation of KS-entropy $\Phi_{2,3}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2, dotted green surface curve shows variation of Lyapunov characteristic exponent of $\Phi_{2,3}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2.

Fig.4. Solid red surface shows the variation of KS-entropy $\Phi_{3,2}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2, dotted green surface curve shows variation of Lyapunov characteristic exponent of $\Phi_{3,2}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2.

Fig.5. Solid red surface shows the variation of KS-entropy $\Phi_{3,3}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2, dotted green surface curve shows variation of Lyapunov characteristic exponent of $\Phi_{3,3}^{\alpha_1,\alpha_2}(x)$, in terms of the parameters α_1 and α_2.

Fig.6. Bifurcation diagram of $\Phi_{2,2}^{\alpha_1,\alpha_2}(x)$: for $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_1 \times \alpha_2 < 4$, it is ergodic and for $\alpha_1 \times \alpha_2 > 4$, it has $x = 1$ fixed point.

Fig.7. Bifurcation diagram of $\Phi_{2,3}^{\alpha_1,\alpha_2}(x)$ and $\Phi_{2,3}^{\alpha_1,\alpha_2}(x)$: for $\alpha_1 > 0, \alpha_2 > 0$ and $\alpha_1 \times \alpha_2 < 6$, it is ergodic and for $\alpha_1 \times \alpha_2 > 6$, it has $x = 1$ fixed point.

Fig.8. Bifurcation diagram of $\Phi_{3,3}^{\alpha_1,\alpha_2}(x)$: for $\alpha_1 > 0 \times \alpha_2 > \frac{1}{9}$ and $\alpha_1 \times \alpha_2 < 9$, it is ergodic, while for $\alpha_1 \times \alpha_2 < \frac{1}{9}$, it has $x = 0$ fixed point and $\alpha_1 \times \alpha_2 > 9$, it has $x = 1$ fixed point.